CHAPTER 4

Solution for Exercise IV

1. (MWG, 6.C.16)

**Solution.** The maximum amount the person is willing to buy the gamble:

\[ 0.5u(w - R_b - y) + 0.5u(w - R_b + x) = u(w), \]

where \( R_b \) is the maximal buying price.

![Figure 1.1. The maximum amount the person is willing to buy the gamble](image)

The minimum amount the person is willing to sell the gamble:

\[ 0.5u(w - y) + 0.5u(w + x) = u(w + R_s), \]

where \( R_s \) is the minimal selling price.

In general, these two prices are different. However, if \( u(\cdot) \) is CARA, then they are the same. In fact, equation (4.1) and (4.2) can be restated as

\[ CE_{w - R_b} = w, \]

\[ CE_w = w + R_s, \]
Figure 1.2. The minimum amount the person is willing to sell the gamble

where $CE_{w-R_b}$ and $CE_w$ are certainty equivalence for (4.1) and (4.2), respectively. The CARA implies that

$$ (w - R_b) - CE_{w-R_b} = w - CE_w, $$

thus

$$ R_b = R_s. $$

2. (MWG, 6.C.20)

Proof.

$$ u(CE) = 0.5u(x + \varepsilon) + 0.5u(x - \varepsilon), $$

where $CE$ is the Certainty Equivalent. Hence

(4.5)  
$$ u'(CE) \frac{\partial CE}{\partial \varepsilon} = 0.5u'(x + \varepsilon) - 0.5u'(x - \varepsilon), $$

(4.6)  
$$ u''(CE) \left( \frac{\partial CE}{\partial \varepsilon} \right)^2 + u'(CE) \frac{\partial^2 CE}{\partial \varepsilon^2} = 0.5u''(x + \varepsilon) + 0.5u''(x - \varepsilon). $$

Thus

$$ \lim_{\varepsilon \downarrow 0} \frac{\partial^2 CE}{\partial \varepsilon^2} = \lim_{\varepsilon \downarrow 0} \frac{0.5u''(x + \varepsilon) + 0.5u''(x - \varepsilon) - u''(CE) \left( \frac{\partial CE}{\partial \varepsilon} \right)^2}{u'(CE)} = \frac{u''(x)}{u'(x)} = -R_a(x), $$

$1$ See MWG (1995) Section 6.C.
since
\[ \lim_{\varepsilon \to 0} \frac{\partial CE}{\partial \varepsilon} = \lim_{\varepsilon \to 0} \frac{0.5u'(x + \varepsilon) - 0.5u'(x - \varepsilon)}{u'(CE)} = 0, \]

and
\[ \lim_{\varepsilon \to 0} u(CE) = u(x). \]

\[ \square \]

3. (Betweenness Axiom)

**SOLUTION.** The Betweenness Axiom\(^2\) only requires that indifference sets be *convex*, i.e., if an individual is indifferent between two lotteries, then any probability mixture of these two is equally good: if \( g \sim g' \), then
\[ \lambda g + (1 - \lambda) g' \sim g, \quad \forall \lambda \in [0, 1]. \]

Essentially, the betweenness axiom is a *substantially weaker* version of the controversial independence axiom. \(^3\)

The axioms 1-4, 6, and the betweenness axiom means that the indifference curves are *straight lines* can be established in the same way as in Chapter 3, exercise (3). Note that we do not use the independence axiom in that exercise, in fact, betweenness axiom is suffices.

Note also that these straight indifference curves *need not* be parallel, because the betweenness axiom imposes restrictions only on straight indifference curves and nothing on the relative positions of different indifference curves.

![Figure 3.1](image-url)

**Figure 3.1.** Betweenness Axiom means the indifference curves are straight lines, but need not be parallel.

\(^2\)See Dekel, E. (1986) for further discussion.

\(^3\)See MWG Exercise 6.B.1\(^A\).
4. (Quadratic v.N-M Utility Function)

a.  
**Solution.** The restrictions are  
\[ u'(w) > 0, \quad u''(w) < 0, \]
so  
\[ b > -4cw; \]
\[ c < 0. \]

\[ \square \]

b.  
**Solution.**  
\[ \mathcal{R}_a(w) = -\frac{4c}{b + 4cw}, \]
so  
\[ \frac{\partial \mathcal{R}_a(w)}{\partial w} = \frac{16c^2}{(b + 4cw)^2} > 0. \]

This means that the quadric utility functions are *unsatisfactory*. Not only do they imply that utility reaches a maximum, they also entail that the absolute degree of risk aversion is *increasing* in wealth, approaching infinity as utility approaches its maximum. Consequently, one is led to the absurd result that the willingness to gamble for a bet of fixed size should decrease as wealth is increased. \[ \square \]

c.  
**Proof.** Before going through the proof, it is worthwhile to consider the intuition of the representation. The expected utility hypothesis suggests that preferences toward gambles can be represented by the *expected value* of a v.N-M utility function  
\[ \mathbb{E}[u(w)], \]
where \( w \) is a random variable that represents the income from an uncertain gamble.

Expected utility in general depends on the form of the function \( u(\cdot) \) and on the distribution of \( w \). Suppose the distribution of \( w \) can be completely characterized by a vector of parameters \( \alpha \). In particular, let \( w \) be distributed on the real line with a P.D.F. \( f(w, \alpha) \). Then\(^4\)
\[ \mathbb{E}[u(w)] = \int u(w) f(w, \alpha) \, dw. \]
The integral on the right-hand side of this equation is a function of \( \alpha \).\(^5\) If we let this integral be represented by  
\[ u(\alpha), \]
then  
\[ u(\alpha) = \mathbb{E}[u(w)] \]
is a valid representation of preferences.

\(^4\)From this subsection through the end of the chapter, we focus on *continuous* monetary variable for convenience.

\(^5\)It is *not* a function of \( w \) since \( w \) is just the variable of integration.
Many problems in the economics of uncertainty are related to the trade-off between the average level of income and its degree of riskiness. Since the mean is a summary measure of average and the variance is a summary measure of risk, it will be particularly convenient to represent preferences by a function of the mean and variance of the income distribution. Unfortunately, this is not always possible, because in general the mean and variance do not completely determine the distribution of a random variable. There are many income streams that have the same mean and variance but different probability distributions. The expected utility associated with these income streams are different. Although \( u(\alpha) \) is a valid representation of preferences, the vector \( \alpha \) generally contains more than two parameters. Thus a utility function that depends only on mean and variance can be at best be viewed as an approximation to expected utility.

There are some special cases, however, when a function involving only the mean and variance of the income distribution can be used to represent preferences. The quadratic utility function in this exercise is a such example.

\[
\mathbb{E}[u(w)] = a + b\mathbb{E}[w] + 2c\mathbb{E}[w^2] = a + b\mathbb{E}[w] + 2c\{\mathbb{E}[w]\}^2 + 2c\text{Var}[w].
\]

\[\square\]

\textbf{d.} 6

\textbf{Solution.} We prove this proposition by an indifference curve in the mean-variance plane.\(^7\) To establish this result, consider two gambles \( g_1, g_2 \in \mathcal{G} \) such that

\[ g_1 \sim g_2. \]

Then, the individual must be indifferent between \( g_1, g_2 \), and a compound gamble

\[ g_q = (q \circ g_1, (1 - q) \circ g_2), \quad q \in [0, 1] \]

where \( q \) denotes the probability of obtaining \( g_1 \), and consequently \((1 - q)\) is the probability of obtaining \( g_2 \).

\[ g_q \sim g_1 \sim g_2. \]

Letting \( \mu_i \) and \( \sigma_i^2 \) denote the mean and variance, respectively, of the distributions corresponding to the gambles \( i = 1, 2, \text{and } q \).

\(^{6}\)Markowitz (1959) demonstrated that if the ordering of alternatives is to satisfy the v.N-M axioms of rational behavior, only a quadratic utility function is consistent with an ordinal expected utility function that depends solely on the mean and variance of the return. Consequently, even if the return for each alternative has a normal distribution, the mean-variance framework cannot be used to rank alternatives consistently with the v.N-M axioms unless a quadratic v.N-M utility function is specified.

\(^{7}\)Our proof gives here follows Baron (1977).
Note that distribution functions preserve the linear structure of gambles (as do P.D.F.'s), so

\[ \mu_q = \int w \left[ qF_1(w) + (1 - q)F_2(w) \right] \]

\[(4.7)\]

\[ = q \int w \, dF_1(w) + (1 - q) \int w \, dF_2(w) \]

\[ = q\mu_1 + (1 - q)\mu_2. \]

\[ \sigma_q^2 = \int \left[ w - q\mu_1 - (1 - q)\mu_2 \right]^2 \left[ qF_1(w) + (1 - q)F_2(w) \right] \]

\[(4.8)\]

\[ = \int \left[ q(w - \mu_1) + (1 - q)(w - \mu_2) \right]^2 \left[ qF_1(w) + (1 - q)F_2(w) \right] \]

\[ = q\sigma_1^2 + (1 - q)\sigma_2^2 + q(1 - q)(\mu_1 - \mu_2)^2. \]

Solving for \( q \) from (4.7) yields

\[ q = \frac{\mu_q - \mu_2}{\mu_1 - \mu_2}. \]

\[(4.9)\]

Substituting (4.9) into (4.8) yields

\[ \mu_q - \left( \frac{\mu_1 - \mu_2}{\mu_1 - \mu_2 + \sigma_1^2 - \sigma_2^2} \right) \left( \mu_q^2 + \sigma_q^2 \right) = \frac{\mu_2\mu_1(\mu_1 - \mu_2) + \mu_2\sigma_1^2 - \mu_1\sigma_2^2}{\mu_1 - \mu_2 + \sigma_1^2 - \sigma_2^2}, \]

or

\[(4.10)\]

\[ \mu_q - \alpha \left( \mu_q^2 + \sigma_q^2 \right) = k, \]

where

\[ \alpha = \frac{\mu_1 - \mu_2}{\mu_1^2 - \mu_2^2 + \sigma_1^2 - \sigma_2^2}; \]

\[ k = \frac{\mu_2\mu_1(\mu_1 - \mu_2) + \mu_2\sigma_1^2 - \mu_1\sigma_2^2}{\mu_1^2 - \mu_2^2 + \sigma_1^2 - \sigma_2^2}. \]

\[ ^8\text{See MWG (1995) p.183.} \]
We can rewrite equation (4.10) again
\begin{equation}
    a + b \mu_q + c \left( \mu_q^2 + \sigma_q^2 \right) = \beta,
\end{equation}
where the parameters \((a, b, c, \beta)\) satisfy
\begin{align*}
    \frac{c}{b} &= -\alpha, \\
    \frac{\beta - a}{b} &= k.
\end{align*}
Equation (4.11) implies that
\begin{equation}
    V(\mu_q, \sigma_q^2) = a + b \mu_q + c \left( \mu_q^2 + \sigma_q^2 \right) = \int (a + bw + cw^2) dF_q(w).
\end{equation}
So the v.N-M utility function that corresponds to \(V(\mu, \sigma^2)\) is the quadratic function
\begin{equation}
    u(w) = a + bw + cw^2.
\end{equation}