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SHUFE Spring 2011

1 Introduction to game theory

1.1 What is Game theory?

1. Why study game theory: Game theory is the branch of microeconomics concerned with the analysis of optimal decision making in competitive situations in which the actions of each decision maker have significant impact on the fortune of the others.

2. What is a game: A game is a formal representation of a situation in which a number of individuals interact in a setting of strategic interdependence. To describe a game, we need to know four things:

   - Players: who is involved (playing in the game)?
   - Rules: How the game is played?
   - Outcomes:
   - Payoffs:

3. How to interpret the concept of Nash equilibrium.

   (a) The central concept of noncooperative game theory is Nash equilibrium. A Nash equilibrium is a profile of strategies or plans, one for each player, such that each player's strategy is optimal for him, given the strategies of the others.

   (b) In most of the early literature the idea of equilibrium was that it said something about how players would play the game or about how a game theorist might recommend that they play the game. However, this interpretation runs into trouble in many cases. For example, how do we interpret mixed strategy Nash equilibrium? How to motivate the refinements of Nash equilibrium?

   (c) Recently, there has been a shift to thinking of equilibria as representing not recommendations to players of how to play the game but rather the expectations of the others as to how a player will play. Further, if the players all have the same expectations about the play of the other players we could as well think of an outside observer having the same information about the players as they have about each other.

1.2 Rationality and common knowledege

1. Rationality Assumption:

2. Common knowledge: A standard assumption is that the game (players, strategies, and payoff functions) is **common knowledge** among players. Common knowledge is an important concept in game theory. A fact is common knowledge among players if each player knows the fact, and each player knows everyone else knows, and each knows everyone else knows everyone else knows, and so on. For example, a handshake is common knowledge between the two persons involved. When I shake hand with you, I know you know I know you know,....., that we shake hand. Neither person can convince the other that she does not know that they shake hand. So, perhaps it is not entirely random that we sometimes use a handshake to signal an agreement or a deal.

3. What if we don't have common knowledge?

   Example 1: Muddy children puzzle

   n children playing together. Each child wants to keep clean, but each would love to see the others get dirty. Now it happens during their play that some of the children, say k ($k > 1$) of them, get mud on their foreheads. Each can see the mud on others but no on his own forehead. No one says a thing. Along comes the father, who says, "At least one of you has mud on your forehead." The father then asks the following question, over and over:"Does any of you know whether you have mud on your own forehead?" Assuming that all the children are perceptive, intelligent, truthful, and that they answer simultaneously, what will happen?

   Example 2: The general's problem

   Two divisions of an army, each commanded by a general, camped on two hilltops overlooking a valley the enemy stays. If both divisions attack the enemy simultaneously they will win the battle, while if only one division attacks it will be defeated. Neither general will attack unless he is absolutely sure that the other will attack with him: a general will not attack if he receives no messages. The general of the first division wishes to coordinate a simultaneously attack (at some time the next day). They can communicate only by means of messengers. Normally, it takes a messenger one hour to get from on encampment to the other and on this particular night, everything goes smoothly. How long will it takes them to coordinate an attack?

2 Strategic form game

2.1 An subjective expected-utility maximization approach

1. Definition: A strategic form game is a tuple $G = (S_i, u_i)_{i=1}^n$, where for all i, S_i is the set of strategies available to player i, and $u_i : \times_{j=1}^n S_j \to \mathbb{R}$ is i's payoff as a function of strategies chosen by all players.

2. An subjective expected-utility approach interpretation

   (a) Each player has a subjective probability distribution over all states of the world– more precisely, the probabilities that her opponents playing s_{-i} for all $s_{-i} \in \times_{j \neq i} S_j$.

   (b) Each player acts as an expected utility maximizer, choosing a strategy that maximizes her expected payoff in the game given the probability distribution over the strategies of her opponents. This is common knowledge.

   (c) The concept of Nash equilibrium imposes a further restriction, player's belief is consistent with the actual play of her opponents.

3. The above interpretation of subjective expected utility-maximization provides a decision theoretical foundation to the traditional definition of Nash equilibrium that each player plays optimally given the other players' equilibrium strategies.

4. An important feature of the subjective expected utility approach is that it does not require randomization on the part of the players. Recall that the traditional interpretation of mixed strategies that assumes players explicitly randomize. The probabilistic

nature of strategies now reflects the uncertainty of other players about a player's choice. Thinking about the traditional Chinese "Scissor-rock-cloth" game.

2.2 Prisoner's dilemma game

The capacity expansion game between Honda and Toyota

Toyota

		Build	Do not build
Honda	Build	16, 16	20, 15
	Do not build	15, 20	18, 18

1. Players: Toyota, Honda.
2. Rules: Two firms simultaneously choose to expand or not.

   Strategies for each firm: Build, Do not build.
3. 4 outcomes: (Build, Build), (Build, Do not build), (Do not build, Build), (Do not build, Do not build).
4. Payoffs:
5. Nash equilibrium of this game: (Build, Build).

2.3 Strategies

1. Definition: A strategy s_i specifies the actions that a player will take under any conceivable circumstances that the player might face.
2. A strategy is a complete contingency plan that says what a player will do at each of her information sets if she is called on to play there
3. A player's strategy may include plans for actions that her own strategy makes irrelevant.

4. One interpretation for the specification of choices at information sets that won't be reached given his strategy is that they are *beliefs* of his opponents about what he would do in case he does not follow his strategy, i.e., the informations were reached. The belief of his opponents is important as their choices at those information sets are based on this belief. Furthermore, what the player's opponents would do at those information sets rationalize his choice at at an upstream information set. Hence, this definition of strategy is not so odd when you interpret it as the way a player determines his strategy.

5. Pure strategy and mixed strategy

   A mixed strategy $\sigma_i \in \Delta(S_i)$ specifies probabilities to two or more pure strategies. For example, the traditional Chinese game, rock, scissor and cloth.

player 2

		scissor	rock	cloth
Player 1	scissor	0,0	-1, 1	1,-1
	rock	1, -1	0,0	-1, 1
	cloth	-1,1	1,-1	0,0

Example 2. A game of "match the coin": Both players, Tom and Jack, choose whether to place the coin Head up or Tail up. Jack wins if two "Head" or two "Tail" appear, and loses otherwise.

Jack

		Head	Tail
Tom	Head	-1, 1	1, -1
	Tail	1, -1	-1, 1

2.3.1 Dominant strategy

1. Dominant strategy: A strategy s_i is dominant for player i if for all $s_{-i} \in S_{-i}$ and $s_i' \in S_i/s_i$

   $$u_i\left(s_i, s_{-i}\right) > u_i\left(s_i', s_{-i}\right).$$

   A strategy is a dominant strategy for a player if it is better than other strategies, no matter what the others will choose.

   In **any Nash equilibrium**, players who have a dominant strategy play the dominant strategy. Thus, it is easy to find the Nash equilibrium of games in which some of the players have a dominant strategy.

2. "Confess" is a dominant strategy for prisoner 1 and prisoner 2.

   Prisoner 1

		Confess	Not confess
Prisoner 2	Confess	-5, -5	0, -10
	Not confess	-10, 0	-1, -1

3. Dominant strategies are rarity rather than norm. There is no dominant strategies in most interesting games.

2.3.2 Dominated strategy

1. A pure strategy $s_i \in S_i$ is weakly dominated if there is another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

   $$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}),$$

   with strict inequality for some s_{-i}.

   A strategy s_i is strictly dominated for player i if there is another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$

   $$u_i\left(s_i', s_{-i}\right) > u_i\left(s_i, s_{-i}\right).$$

   A strategy is strictly **dominated** when the player has another strategy that gives her a higher payoff no matter what the other player plays.

2. Allowing mixed strategy:
   - A strategy σ_i is strictly dominated for player i if there is another strategy $\sigma_i' \in \Delta(S_i)/\sigma_i$ such that for all $\sigma_{-i} \in \Delta(S_{-i})$

     $$u_i\left(\sigma_i, \sigma_{-i}\right) < u_i\left(\sigma_i', \sigma_{-i}\right).$$

   - A pure strategy s_i is strictly dominated for player i if and only if there exists $\sigma_i \in \Delta(S_i)$ such that

     $$u_i\left(s_i, s_{-i}\right) < u_i\left(\sigma_i, s_{-i}\right)$$

     for all s_{-i}.

3. A mixed strategy that assigns positive probability to a pure strategy that is strictly dominated is also strictly dominated.

4. Modifies capacity expansion game between Toyota and Honda

   Toyota

		large	small	not build
Honda	large	0, 0	12, 8	18, 9
	small	8, 12	16, 16	20, 15
	not build	9, 18	15, 20	18, 18

   "large" is a dominated strategy for both firm as it can do better by choosing "small", regardless of what the other firm is going to do.

5. A player will not play a dominated strategy in Nash equilibrium.

2.4 Nash equilibrium

2.4.1 Definition and interpretation

1. Definition: A strategy profile $s^* = (s_1^*, ..., s_n^*)$ is a **Nash Equilibrium** (NE) if for all $i \in N$ and for all $s_i' \in S_i$

   $$U_i\left(s_i^*, s_{-i}^*\right) \geq U_i\left(s_i', s_{-i}^*\right).$$

2. The central concept of noncooperative game theory is Nash equilibrium. A Nash equilibrium is a profile of strategies such that for each player in the game, given the strategy chosen by the other players, the strategy is a best response for the player, that is, the strategy gives the player the highest payoff.

3. In early literature, the idea of equilibrium was that it said something about how players would play the game, or how game theorists would recommend them to play the game. More recently, game theorists look to an alternative interpretation. According to this interpretation, an equilibrium is taken as every player making optimal choice given the belief about how opponents would play the game given their respective beliefs.

4. While the first interpretation of the equilibrium can be problematic in case of mixed strategy equilibrium, the second interpretation can accommodate mixed strategies without any trouble. In this scenario, the mixed strategy of a player does not represent a conscious randomization on the part of that player, but rather the uncertainty in the minds of the others as to how that player will act. Hence, the second interpretation of Nash equilibrium has become the preferred interpretation among game theorists.

   Thus the focus of the equilibrium analysis becomes, not the choices of the players, but the assessments of the players about the choices of the others. The basic consistency condition that we impose on the players' assessments is this: A player reasoning through the conclusions that others would draw from their assessments should not be led to revise his own assessment.

2.4.2 Strategic Stability of NE

1. Being a NE is a necessary condition for an obvious way to play the game, if an obvious way to play the game exists. But

   - Being NE is not sufficient for a strategy profile to be the obvious way to play a given game.
   - Not every game admits an obvious way to play the game

2. Some questions to be answered:

- How can we refine NE, the necessary condition to get the prediction of the game, an obvious way to play the game.

  NE can involve weakly dominated strategies, we should add to our necessary condition that the solution should be a NE in strategies that are undominated, even weakly

- What are the means by which we are to identify "obvious way to play a game?"

- What can one say about games that do not admit a "solution"

  When the game does not admit an "obvious way to play," looking at its NE can give precisely the wrong answer. The concept of NE is of no use when the game admits no "solution"

2.4.3 Find NE in two-player games

In games with dominant and dominated strategies

1. Most games have finite and **ODD** number of NE.

   Example:

	L	M	R
T	3, 3	0, 2	3, 0
B	0, 0	3, 2	0, 3

2. If both players have a dominant strategy, then playing dominant strategies is the unique Nash equilibrium in the game.

3. If one player has a dominant strategy, this strategy will be this player's NE strategy. The other player's NE strategy is the best response to her opponent's dominant strategy.

4. If players have dominated strategies, delete the dominated strategies from the game and work with a smaller game.

   Example

Player 2

		A	B	C
Player 1	A	5, 8	15, 10	10, 5
	B	10, 15	20, 9	15, 0
	C	20, 20	10, 10	10, 8

5. NE must be a mutual best-response, that, given player 1 plays NE strategy, player 2 can not do better by playing some other strategies, similarly, given player 2 plays this strategy, player 1 can not do better by changing strategies. A NE is a strategy profile in which both players are playing the best-response given the other player's strategy.

2.4.4 Three player game

1. A simple example

Player 2

		U	D
Player 1	A	(1, 1,0)	(2, -2, 5)
	B	(1,-2, -1)	(0, 3, 1)

Player 3 plays L

Player 2

		U	D
Player 1	A	(1, 1,-2)	(2, -2, 5)
	B	(2, 2, -1)	(2, 3, 7)

Player 3 plays R

Pure strategy NE in this game: (A, U, L), (B, D, R).

2. Another example

Player 2

		U	V	W
	L	3, 0, 2	2, -1, 0	1, -2, 0
Player 1	M	3, 2, 1	1, 4, -1	0, 0, -2
	R	1, 1, 10	0, 2, 1	-2, 0, 3

Player 3 plays A

Player 2

	U	V	W
L	2, 1, 1	3, 0, 0	2, -2, -1
M	5, 4, 2	1, 3, 4	3, 0, -2
R	1, 1, 1	0, 2, 0	-2, 0, 2

Player 3 plays B

Player 2

Player 1

	U	V	W
L	2, 1, -1	3, 0, -1	2, -2, -3
M	5, 4, -1	1, 3, -2	3, 0, -4
R	1, 1, -10	0, 2, -1	-2, 0, -2

Player 3 plays C

Player 1's dominated strategy R, player 2's dominated strategy W. Player 3's dominated strategy C. The pure strategy NE in this game (L, U, A), (M, U, B).

2.5 Iterated Deletion of Strictly Dominated Strategies

1. Now in situations where players do not have a chance to talk or where there is no history to rely on, Nash equilibrium may not be a good prediction. So in this case, we may prefer a solution concept that does not make strong assumptions about players knowing what each other is going to do.

2. A rational player should never choose a strictly dominated strategies because there exists another strategies that is strictly better. Note that if a player has a dominant strategies, then all other strategies are dominated.

3. Consider the following game (Gibbons pp. 6) :

   Player 2

   Player 1

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

4. Note that R is strictly dominated by M for Player 2. Now, if Player 1 knows that Player 2 is rational, then Player 1 knows that Player 2 will never choose R. If R is eliminated, then D becomes dominated by U. Now, if Player 2 knows Player 1 knows that Player 2 is rational, then Player 2 knows that Player 1 will not choose D. In that case, Player 2 should choose M.

5. If it is common knowledge that both players are rational, we can continue this process indefinitely. The set of strategies that survive this process is called **rationalizable** for two-player games.

6. Note that (U,M) is also the unique Nash equilibrium. In general, any strategy is part of a Nash equilibrium is rationalizable. But the converse is not true. Iterated deletion of strictly dominated strategies is strictly weaker than Nash.

7. Note that Nash and IDSDS are based on different logic. IDSDS does not require that the players know that the equilibrium is going to be played, so it requires less coordination. However, common knowledge of rationality is itself a very strong assumption.

8. Rationalizable strategies

    (a) A strategy σ_i is a best response for player i to her rivals' strategies σ_{-i} if

    $$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}).$$

    for all $\sigma_i' \in \Delta(S_i)$

    (b) Strategy σ_i is never a best response if there is no σ_{-i} for which σ_i is a best response.

    (c) The strategies in $\Delta(S_i)$ that survive iterated deletion removal of strategies that are never a best response are known as player i's rationalizable strategies.

Player 2

		b_1	b_2	b_3	b_4
	a_1	0, 7	2, 5	7, 0	0, 1
Player 1	a_2	5, 2	3, 3	5, 2	0, 1
	a_3	7, 0	2, 5	0, 7	0, 1
	a_4	0, 0	0, -2	0, 0	10, -1

2.6 Existence of NE

1. Theorem 1 (Theorem 7.2, Jehle and Reny). Every finite strategic form game has at least one NE.

    Proof. See Jehle and Reny pp. 278.

2. Theorem 2. NE exists if the strategy set of each player is a compact and convex subset of an Euclidean space and if the utility function of each player is continuous in the strategy profile and quasi-concave in one's own strategy.

   Proof. Step 1: the maximizer

   $$b_i(\sigma_{-i}) = arg \max_{\sigma_i \in \Delta(S_i)} \ u_i(\sigma_i, \sigma_{-i})$$

   is nonempty, convex-valued and upper hemicontinuous.

   Step 2: by Kukutani's fixed point theorem, a non-empty, convex-valued upper hemicontinuous correspondence $b_i(\sigma_{-i})$ mapping from $\Delta(S)$ to itself, there must exists a fixed point.

3 Normal-form perfect equilibrium

1. The problem with NE is, that, many games have multiple equilibria. The natural question then arises, can we go any further and rule out any equilibria as self-enforcing assessment of the game. Indeed, on occastion irrational assessments by two different players might each make the other look rational.

2. As an example, consider the following game

	b_1	b_2
a_1	3, 3	0, 0
a_2	-5, -5	0, -5

   But is (a_2, b_2) a good prediction of the game? Not likely.

3. An *ϵ-perfect equilibrium of the normal form game* is a totally mixed strategy $\sigma \equiv (\sigma_1, \ldots, \sigma_N)$, if for all i and for all $s_i, s_i' \in S_i$,

   $$u_i(s_i, \sigma) > u_i(s_i', \sigma) \qquad \text{then} \qquad \sigma_i(s_i') \leq \epsilon.$$

4. A *perfect equilibrium* of a normal form game is a limit ($\epsilon \to 0$) of ϵ-perfect equilibria.

5. For two-player game, any NE in which no player plays dominated strategies is perfect.

6. For more than two-player game, the above statement is not true. There are NE with no players playing dominated strategies that is not perfect.

   Consider the 3-player entrant-incumbent example.

   Firm E2

Firm E1		Accept	Decline
	OI	0, 0, 3	0, 0, 3
	OO	0, 0, 3	0, 0, 3
	EI	-1, 0, 2	-1, 0, 2
	EO	-1, 0, 2	-1, 0, 2
	PI	1, 1, -2	-1, 0, 2
	PO	1, 1, -2	0, 0, 3

   firm I fight

   Firm E2

Firm E1		Accept	Decline
	OI	0, 0, 3	0, 0, 3
	OO	0, 0, 3	0, 0, 3
	EI	2, 0, 1	2, 0, 1
	EO	2, 0, 1	2, 0, 1
	PI	4, 4, 0	2, 0, 1
	PO	4, 4, 0	0, 0, 3

   I accommodate

   - In the NE (PO, A, Accommodate), no player plays dominated strategy.
   - But (PO, A, Accommodate) is not a perfect equilibrium.

7. Perfect equilibrium does not eliminate all unreasonable outcomes in some games. Adding a domianted strategy may enlarge the set of perfect equilibria.

   Consider the following example.

	L_2	M_2
L_1	1, 1	0, 0
M_1	0, 0	0, 0

	L_2	M_2	R_2
L_1	1, 1	0, 0	-1, -2
M_1	0, 0	0, 0	0, -2
R_1	-2, -1	-2, 0	-2, -2