An approach to VaR for capital markets with Gaussian mixture

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Abstract

An approach to VaR (value-at-risk) for capital markets is proposed with Gaussian mixture. Considering the impacts of the components in a Gaussian mixture, an approach to VaR for capital markets is proposed to describe risk structure in capital markets. This approach can be programmed in parallel. Empirical computation of VaR for China securities markets and the Forex markets are provided to demonstrate the proposed method.

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1. Introduction

The computational framework for value-at-risk (VaR) is a useful methodology for estimating the exposure of a given portfolio of securities to different
kinds of risk inherent in financial environment. One driving force behind the popularity of this technique is the release to the studies of many researchers and the documents of JP Morgan and Basle Committee on Banking Supervision [1,2]. Another one is so-called random walks hypothesis about capital markets [3]. However, empirical evidence showed that the returns are actually fat tailed and that there exists non-random walks in capital markets and thus suggested that the assumption that returns of financial assets are normally distributed is inappropriate [4,5]. VaR calculated under the normal assumption underestimates the actual risk [6–9]. Zangari treated two components in a Gaussian mixture to fit the fat of financial assets [6]. Venkataraman provided an estimation techniques for value-at-risk in a mixture of normal distributions [7]. Hull and White suggested to use alternative distributions, such as a mixture of two normal distributions, to model the return of financial assets [8]. Li made use of statistics such as volatility, skewness and kurtosis to capture the extreme tail [9]. It is seen that these methods paid attention to few components in a Gaussian mixture. However, empirical studies showed that the number of components is greater than two in a Gaussian mixture for the returns of financial assets [10–12].

In this paper, an approach to VaR for capital markets with Gaussian mixture takes into account of not only non-normal distribution but also the impacts of the components in a Gaussian mixture. In Section 2, we propose an approach to VaR for capital markets with Gaussian mixture that uses lots of the components in a Gaussian mixture to describe risk structure in capital markets. In Section 3, we provide empirical computation of VaR for China securities markets and the Forex markets to demonstrate the proposed method. In Section 4, some discussions are given.

2. The computational framework for value-at-risk

In order to compute VaR for capital markets on the condition of non-random walks, we use Gaussian mixture to measure the risk of financial assets in capital markets. It is well known that Gaussian mixture can be used not only to fit returns of financial assets but also to capture non-normality of financial assets movements [11,12]. So, we pay attention to computational framework of VaR for capital markets with Gaussian mixture.

Let $S_t \in \mathcal{R}$ be the financial asset prices series (stock, index, or exchange rate) and $\mathcal{S} = \{S_{-1}, S_0, S_1, \ldots, S_{n-1}\}$. Define returns of financial assets time series as

$$r_t = \log(S_t/S_{t-1}) \quad \forall S_t \in \mathcal{S} \quad \forall t \in \{0, \ldots, n - 1\}$$

and let $\mathcal{R} = \{x_0, x_1, \ldots, x_{n-1}\}$ and $S_{-1} = S_0$.

Suppose that a component distribution $f_k(r|\mu_k, \sigma_k^2)$ is Gaussian, i.e. $N(\mu_k, \sigma_k^2)$, named this component as $\omega_k$. Let $K$ be the number of components,
\( \{ \mu_k, \sigma^2_k \}_{k=1}^K \) be the set of unknown parameters, and \( p_k \) be mixing proportion of the \( k \)th component \( \omega_k \). Gaussian mixture can be expressed as

\[
    f(r|\Theta) = \sum_{k=1}^K p_k f_k(r|\mu_k, \sigma^2_k),
\]

(2)

by a sum of distributions or equivalently denoted as

\[
    (r_t, \eta) = \begin{cases}
        r_t|\omega_1 \sim N(\mu_1, \sigma^2_1), & p(\eta = \omega_1) = p_1, \\
        \cdots \\
        r_t|\omega_k \sim N(\mu_k, \sigma^2_k), & p(\eta = \omega_k) = p_k, \\
        \cdots \\
        r_t|\omega_K \sim N(\mu_K, \sigma^2_K), & p(\eta = \omega_K) = p_K,
    \end{cases}
\]

(3)

by a pair of random variables, where \( K \geq 1, \quad 0 < p_k \leq 1, \quad \sum_{k=1}^K p_k = 1 \) and \( \Theta = \{ p_1, \mu_1, \sigma^2_1; \ldots; p_K, \mu_K, \sigma^2_K \} \) and \( \eta \) is a latent indicator, i.e. \( \eta = \omega_k \) if and only if \( r_t \in \omega_k \), with certain probabilistic structure.

The unknown parameters, \( \Theta \) and \( K \), can be estimated by the issues [12–14]. The empirical evidence has to show that the number of components, \( K \), in a Gaussian mixture is greater than two [10–12] and conduces to an approach to VaR for capital markets with Gaussian mixture. This approach differs from the methods suggested by the researchers [6–9] in the impacts of components in a Gaussian mixture. It is the impacts of components in a Gaussian mixture that bring on various risky levels and make lots of the components to describe risk structure in capital markets.

General speaking, VaR is a single, summary and statistical measure of possible portfolio profit and losses due to random walks in capital markets, which corresponds to the low \( \alpha \) quantile of a distribution of returns. For any significance level \( \alpha \in (0, 1) \), a distribution function \( F(r) \) for random variable \( r \), VaR at the probability level \( \alpha \) is defined as

\[
    \text{VaR} = F^{-1}(\alpha) = \inf \{ r | F(r) \geq \alpha \},
\]

(4)

where, \( F(r) \) is Gaussian distribution. It is this so-called normality that stirs up a deviation of risk measurement from the real world.

Considering the impacts of the components in a Gaussian mixture, an approach to VaR for capital markets is proposed. According to Gaussian mixture and VaR correspondingly to Eqs. (2)–(4), it is obvious that the following theorems hold true.

Theorem 1. For the given significance level \( \alpha \in (0, 1) \), the respective VaRs of a Gaussian mixture and its \( k \)th component can be expressed as

\[
    \begin{cases}
        \sum_{k=1}^K p_k \int_{-\infty}^{\text{VaR}_k(\alpha)} f(r|\mu_k, \sigma^2_k) \, dr = \alpha, \\
        \int_{-\infty}^{\text{VaR}_k(\alpha)} f(r|\mu_k, \sigma^2_k) \, dr = \alpha.
    \end{cases}
\]

(5)
The relationship to the various latent risks of a financial asset

\[
\min_{1 \leq k \leq K} \{ \text{VaR}^k_{mx}(z) \} \leq \text{VaR}_{mx}(z) \leq \max_{1 \leq k \leq K} \{ \text{VaR}^k_{mx}(z) \}
\]

always holds true.

The VaR_{mx}(z) and \text{VaR}^k_{mx}(z) calculated according with Eq. (5) can describe the movement of price behaviors in capital markets at the given significance level \( x \).

**Theorem 2.** For the given VaR, the respective significance levels on a Gaussian mixture and its kth component can be represented as

\[
\begin{align*}
\alpha_{mx} &= \sum_{k=1}^{K} p_k \int_{-\infty}^{\text{VaR}} f(r|\mu_k, \sigma_k^2) \, dr, \\
\alpha^k_{mx} &= \int_{-\infty}^{\text{VaR}} f(r|\mu_k, \sigma_k^2) \, dr.
\end{align*}
\]

The relationship to the various latent confidences in a financial asset

\[
\min_{1 \leq k \leq K} \{ \alpha^k_{mx} \} \leq \alpha_{mx} = \sum_{k=1}^{K} p_k \alpha^k_{mx} \leq \max_{1 \leq k \leq K} \{ \alpha^k_{mx} \}
\]

always keeps true.

The \( \alpha_{mx} \) and \( \alpha^k_{mx} \) evaluated according with Eq. (7) can feel the assurance on capital markets at the given VaR.

The sum and substance of Theorems 1 and 2 can be also stated that for financial asset \( r_t \), which belongs to the kth components \( \omega_k \) in a Gaussian mixture, its VaR follows that

\[
(\text{VaR}(r_t, z), \eta) = \begin{cases} 
\text{VaR}_{mx}^1(z), & p(\eta = \omega_1) = p_1, \\
\cdots & \\
\text{VaR}_{mx}^k(z), & p(\eta = \omega_k) = p_k, \\
\cdots & \\
\text{VaR}_{mx}^K(z), & p(\eta = \omega_K) = p_K,
\end{cases}
\]

where \( \eta \) is a latent indicator, i.e., \( \eta = \omega_k \) if and only if \( r_t \in \omega_k \), with certain probabilistic structure \( \{p_k\}_{k=1}^{K} \) and \( x \) is significance level.

As a result, such things as a Gaussian mixture, Eqs. (2) and (3), Theorems 1 and 2 constitute an approach to VaR for capital markets that use lots of the components in a Gaussian mixture to describe the risk structure in capital markets as shown in Table 1 and Eq. (9).
3. Empirical computation

In order to demonstrate the proposed method, we provide the empirical computation of VaR for China securities markets, i.e., Shanghai index and Shenzhen index, and the Forex markets, i.e., Interbank Rate of US dollar to Deutsche Mark, from 1996 to 2001. The unknown parameters, $H$ and $K$, are estimated by the EM algorithm based on KS test. The tests showed that the number of components in a Gaussian mixture is more than two [12,14]. Based on Gaussian mixture, the behaviors of price movement and the psychologies of investors in capital market are showed in Table 2.

With regard to the behaviors of price movement in capital markets, it is obvious that for the given significance level $z = 0.99$, VaRs on the condition of normal distribution, i.e., $\text{VaR(Shanghai)} = 4.57109$, $\text{VaR(Shenzhen)} = 5.14056$ and $\text{VaR(US2DEM)} = 1.18379$, differ from those in a principal component [12], i.e., $\text{VaR}_{mx}^1(\text{Shanghai}) = 2.34484$, $\text{VaR}_{mx}^1(\text{Shenzhen}) = 2.781$ and $\text{VaR}_{mx}^1(\text{US2DEM}) = 1.91533$, but also those in a Gaussian mixture respectively, i.e., $\text{VaR}_{mx}(\text{Shanghai}) = 5.63919$, $\text{VaR}_{mx}(\text{Shenzhen}) = 6.04621$ and $\text{VaR}_{mx}(\text{US2DEM}) = 1.44815$. The differentiae show that VaR on the normal assumption is biased from the real world in contrast to a Gaussian mixture and then the behaviors of price movements in China securities markets were more risky than in the Forex markets, which resulted from the emerging markets in China.

Referring to the psychologies of investors in capital markets it also be apparent that for the given VaRs on the condition of normal assumption, i.e., $\text{VaR(Shanghai)} = 4.57109$, $\text{VaR(Shenzhen)} = 5.14056$, and $\text{VaR(US2DEM)} = 1.18379$, the significance level $z = 0.9900$ in the Gaussian distribution is different from one in a principal component, i.e., $\gamma_{mx}^1(\text{Shanghai}) = 0.999967$, $\gamma_{mx}^1(\text{Shenzhen}) = 0.999967$ and $\gamma_{mx}^1(\text{US2DEM}) = 0.921415$, and another one in a Gaussian mixture respectively, i.e., $\gamma_{mx}(\text{Shanghai}) = 0.981153$, $\gamma_{mx}(\text{Shenzhen}) = 0.981471$ and $\gamma_{mx}(\text{US2DEM}) = 0.966648$.
It seems that the significance level \( \alpha \) on the condition of normal assumption is unreliable in comparison of Gaussian mixture and then the psychologies of investors in China securities markets were more hazardous than in the Forex markets, which stemmed from irrational investors in China securities markets.

### 4. Discussion

On above the statement and the empirical computation, it can be seen that this approach to VaR for capital markets with Gaussian mixture has advantage of the description of various risk levels. Illustrated by our empirical computation, an approach to VaR for capital markets with Gaussian mixture is different not only from one with the \( k \)th component in a Gaussian mixture but also from another one with Gaussian distribution and it seems that there will exist mistake without of Gaussian mixture.

It should be pointed out that there exists a key problem to be resolved that the component in a Gaussian mixture that returns of financial assets in trading day belongs to must be determined in practice.
References