Seignorage, Productive Government Spending and Growth in a Lucasian General Equilibrium Model

Hans–Martin Krolzig*, a,b, Don I. Asoka Wöhrmannc

a Institute of Economics and Statistics, University of Oxford, Oxford OX1 3UL, UK
b Nuffield College, Oxford OX1 1NF, UK.
c Department of Operations Research and Systems Theory, Vienna University of Technology, Austria.

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Abstract

This paper examines the effects of fiscal and monetary policies in a two-sectorial endogenous growth model with money in the utility function. By focusing on the linkage between monetary and fiscal policy due to the government budget constraint, our model deviates from the literature. We show that the impacts of monetary policy heavily depend on the allocation of the seignorage. While money is superneutral, long-run economic growth depends on the level of government expenditures in the human-capital production sector. Therefore an increase in the rate of money supply growth has real effects if and only if the resulting seignorage is used for human capital investments.

Keywords: Endogenous growth; Human capital, Government spending, Money; Neutrality; Super-neutrality.

JEL classification: O42; E60; E00.

1 Introduction

Understanding the factors contributing to long-run growth is a major interest in economics for theorists, empiricists, and policy designers alike. Economic thinkers through the ages, from Adam Smith to Thomas Malthus to Joseph Schumpeter, have contemplated the forces that contribute to economic growth. That said, even though economic growth has emerged as an explicit goal of government policy since the Second World War, understanding economic growth has not been the first priority of the economics profession.

In the standard neoclassical growth model steady-state growth is purely determined by the exogenous growth rate of labor supply and the rate of technical progress. Hence, these models are unable to explain the effect of government policies on the rate of economic growth. Fiscal policy can affect only

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Address for correspondence: Institute of Economics and Statistics, University of Oxford, St Cross Building Manor Road, Oxford OX1 3UL, UK; phone: +44-1865-271085; fax: +44-1865-271094; e-mail: hans-martin.krolzig@nuf.ox.ac.uk.
the per capita levels of capital and income in the long run. Due to this result, the conventional wisdom has been that different taxation and expenditure policies can be crucial determinants of the level of output but are unable to have a substantial effect on the growth rate. This traditional view, however, contrasts with the predictions of the most recent developments in the theory of economic growth. Yet, economists will most often maintain that fiscal policy constitutes a crucial determinant of the growth performance of a particular country. Consequently, growth miracles, but also stagnation effects of various tax and other fiscal policies, have been persistently emphasized in the growth theory literature during the last three decades.

Thus, the classic works in the new growth literature, including Romer (1986, 1990), Lucas (1988), Jones & Manuelli (1990) and Rebelo (1991), all stress that long run growth in output per capita must be explained endogenously.

Several authors have recently begun to examine the role of public policy in generating long-run growth. The recent developments have significant policy implications since public policy in dynamic general equilibrium models may influence long-run growth rates and welfare. For instance, Jones and Manuelli (1990), King & Rebelo (1990), Lucas (1990), and Rebelo (1991), Jones et al. (1993), Pecorino (1993), Devereux & Love (1995), and Greiner (1996) examine whether national fiscal (tax) policies could explain the observed disparity in growth rates between countries. Most of these models attempt to transform the temporary growth effects of fiscal (tax) policy implied by the neoclassical model into permanent growth effects. However, most of these models isolate the effects of taxation on long-run growth by assuming that government expenditures do not affect households’ preferences or production technology. Another weakness is that these fiscal endogenous growth models neglect the allocative effects of different government expenditure policies on growth.

The first article to receive attention within this field government and growth was written by Barro (1990), who presents a simple model of endogenous growth, where the interaction between private and public capital is elegantly captured. Barro (1990) explores a link between productive government expenditures and growth. In his simple one-sector endogenous growth model, the government uses income tax revenues to finance productive government expenditures. Public services are considered to be inputs to private production and it is this complementarity between public and private capital that creates a potentially positive linkage between government services and growth in the model. Barro shows that expenditures on services which enhance the productivity of the private sector increase the rate of growth. In his model, Barro shows that even though government spending has a positive impact on growth by enhancing the marginal productivity of private capital, this effect may be offset by the negative effect of distortionary income taxation. Government services can be thought of as a flow in the sense that the government buys a flow of output from the private sector and makes it available to all households. These services correspond to the input that matters for private production. As long as the government and the private sector have the same production function, the result would be the same if the government bought private inputs and engaged in production itself, instead of purchasing only final output from the private sector. The point of including public capital as a separate argument of the production function is that private inputs are not always close substitutes for public inputs and user charges have proven difficult for private producers to implement. Prime example are public services such as national defense and the maintenance of law and order. The government therefore has a role to play as a provider of certain goods, as argued by Devarajan et al. (1996). Exactly how one should define public capital can be disputed.

If revenues are used to finance government services that have no effect on productivity, or if they are
wasted by bureaucrats, then the consumption growth rate will decrease\(^1\). Barro assumes that government expenditure on investment is a flow variable, whereas Arrow & Kurz (1970) suppose that only the stock of public capital positively influences the productivity of private capital\(^2\).

Futagami et al. (1993) extend Barro’s one-sector model and integrate the Barro - model with the Arrow and Kurz assumption that government investment does not influence the macroeconomic production function directly but only indirectly through the stock of public capital which, for its part, positively influences the marginal productivity of private capital. However, it should be noted that Barro (1990) and Futagami et al. (1993) neglect the importance of the allocation of government expenditures for growth.

In a two-sectorial model, Lin (1994) shows that the allocation of a given level of government expenditures between the production sectors can be important as well. But the steady-state equilibrium exhibits a continuously decreasing government share of GNP. However, these models concentrate on the change in the size of government expenditures and taxes given a balanced budget. In particular, they are neglecting the dependence of monetary and fiscal policy due to the government budget constraint.

\(\text{Bitte nimm was Du so für die Einleitung so brauchst!!} \)

Since Tobin, many researches are addressed their studies to the question, how does a change in monetary policy can affect the growth rate of the economy?

The relationship between money and growth has been one of the classical topics in monetary theory. The work of Tobin (1965), Sidrauski (1967b), and Dornbusch & Frenkel (1973) has shown that in models with a fixed saving rule an increase in money creation induces a higher steady-state capital-labour ratio. However, these works have been criticised because such savings rules are ad hoc. Such criticism, for instance by Johnson (1969), has led to growth models with individual behaviour based on explicit utility maximisation. The first explicit formulation of money in an optimising growth model is due to Sidrauski (1967a). His model, yielding long-run superneutrality, joins Patinkin’s (1965) work on currency in the utility function and Ramsey’s (1928) analysis of optimal savings behaviour. The issue of the superneutrality of money was pointed out by Tobin (1965), whose analysis predicts a positive correlation between money supply growth and capital stock. Hence, variations in the monetary growth rate have real effects in the long run and are not only reflected by changes in the rate of inflation. An inverse relationship between money growth and capital was derived in Brock (1974) when the supply of labor is endogenous. Stockman (1981) presented a model in which money growth and capital are negatively related when a cash-in-advance (CIA) constraint applies to both consumption and investment, while money is superneutral in his model when only consumption is subject to the CIA constraint. Aschauer and Greenwood (1983) have examined the effects of inflation on the effective price of goods, and its influences on labor supply decisions.

More recent studies on money and growth once more investigate the effects of inflation in an endogenous growth framework. While De Gregorio (1993) contemplated different channels through which

\(^{1}\text{Of course, if the government services are desired by individuals, the implications for welfare remain ambiguous. Another interesting conclusion of Barro’s paper is the proof that maximising the national growth rate is equivalent to maximising social welfare.}\)

\(^{2}\text{Later, in the context of the government budget constraint, we will discuss this aspect of government investment much more intensively.}\)
inflation affects the growth process. De Gregorio emphasizes the effect of inflation on the rate of investment and on the productivity of investment, respectively. He considers the inflation as a part of a government’s finance problem, where high rates of inflation are the result of inefficiencies in the national tax system.

A negative effect of inflation on growth was established by Gomme (1993) in a stochastic, dynamic framework with endogenous growth and money. In Gomme’s neoclassical setting with human capital production, requiring both physical capital and labor input, is the engine of growth. For his negative relationship between money and growth, the endogenous labor supply is crucial in producing Gomme’s results.

Aschauer and Greenwood (1983) have examined the effects of inflation on the effective price of goods, and its influences on labor supply decisions.

A very prominent work by Jones et al. (1995) study the relationship between money growth and output growth in an economy. They show that in some models of endogenous growth, inflation has direct effects on the fundamental growth rate of the economy.

A very recent work by Huo (1997) examines in a dynamic two-sector model in which money is introduced through a CIA constraint. Another feature of his paper is that a perfectly anticipated inflation changes the relative price between consumption goods, as they are differentially subject to the CIA constraint. In contrast to Stockman (1981) Huo concludes that inflation can either increase or decrease capital accumulation, even though only consumption goods is subject to the CIA constraint.

We want to exploit recent developments in the endogenous growth theory to extend those mechanisms to environments where growth can be sustained through the human capital accumulation as already discussed by Lucas (1988).

Our model deviates from the existing literature by focusing on the linkage between monetary and fiscal policy due to government budget constraint and the allocation of government expenditures. The take-off point for our analysis is the Lucas’ (1988) endogenous two-sector growth model. In order to study the effects of monetary and fiscal policies on long-run growth, we consider the Lucas framework from a new angle. Our model incorporates money into the utility function and government expenditures into the market production and the human-capital production functions.

The paper is organised as follows. Section 2 describes the economic environment of this model. The competitive equilibrium is characterised in section 3. In section 4 we describe the balanced growth path and in section 5 we analyse the effects of fiscal and monetary policy on inflation and long-run growth. Our conclusions are presented in section 6.

2 The Economic Environment

2.1 Money in the Utility Function

Money is introduced by assuming that in addition to consumption, utility is derived from the flow of services derived from real money holdings. The utility functional to be maximised is then

$$U(0) = \int_{0}^{\infty} u(c, m)e^{-\rho t} dt,$$

where \(c\), \(m\), and \(\rho\) are consumption, the real money balance per-capita, and the subjective discount rate, respectively. We assume that the population, which corresponds to the number of workers and

\(3\) Feenstra (1986) shows that the procedure of entering money into the utility function is formally equivalent to entering liquidity costs into the budget identity of the representative household, so that Sidrauski’s approach is not as restrictive as it seems at first sight.
consumers, is normalised to one and that the instantaneous utility function is of the CES-form

\[ u(c, m) = \frac{1}{1 - \sigma} \left\{ [(1 - \alpha)c^\varepsilon + \alpha m^\varepsilon]^{\frac{1}{\sigma}} \right\}^{1 - \sigma}, \quad 0 < \alpha < 1, \varepsilon < 1. \]  \tag{2}

Equation (1) can be considered as a generalised reduced utility function, where utility depends on consumption and leisure while being subject to a transaction technology. In McCallum (1989), for instance, leisure is identified as non-shopping time and money is a substitute for leisure since it facilitates transactions\(^4\).

Hence, we assume that agents maximise a per-capita utility function of the form

\[ \int_0^\infty e^{-\rho t} \frac{1}{1 - \sigma} \left\{ [(1 - \alpha)c(t)^\varepsilon + \alpha m(t)^\varepsilon]^{\frac{1}{\sigma}} \right\}^{1 - \sigma} dt, \]  \tag{3}

where \( \alpha \) is relative utility share of real money holding and \( \sigma \) is the inverse of the elasticity of intertemporal substitution, while \( \frac{1}{1 - \varepsilon} \) denotes the intratemporal elasticity of substitution between consumption and money.

2.2 Technology

The assumed production technology extends the approaches found in Lucas (1988), King & Rebelo (1990), and Rebelo (1991), by incorporating productive government services. It can also be considered as an extension of Barro (1990) by integrating a sector of human capital production into a one-sector endogenous growth model where productive government services are factors of the private production technology. Let \( g \) be the quantity of public services provided to each producer-household without any user charges, and further assume that the government uses tax and seignorage revenues to provide infrastructure investments in the physical production sector, \( g_k \), and human capital investments, \( g_h \), to the human capital sector\(^5\). The government services as an input to private production play a productive role, which creates a substantive positive linkage between government and growth.

But as noted above, the take-off point of our analysis is Lucas' (1988) endogenous two-sector growth model. Thus, consider an economy in which there are two productive activities: market or physical output production and human capital production. Each producer-household has access to the following physical production function

\[ y = f(k, uh, g_k) = Ak^{\beta_1}g_k^{\beta_2}(uh)^{1 - \beta_1 - \beta_2}, \]  \tag{4}

where \( y \) is output per worker, \( k \) is capital per worker and \( uh \) can be interpreted as effective labor input, which depends on the worker’s human capital and on his non-leisure allocation decision, namely how much time he wants to devote to physical production. The exogenously given technology level \( A \) is assumed to be constant.

The production technology (4) exhibits constant return to scale (CRS) in the three inputs together, following Barro (1990), the return to private accumulation is diminishing, \( \beta_1, \beta_2 \in (0, 1) \).

but diminishing returns to private inputs separately\(^6\). Along with Barro (1990), we abstract from externalities associated with the use of public services. In contrast to Lucas (1988), externalities from human capital do not appear (\( \gamma = 0 \)).

\(^4\)Cash-in-advance constraints (CIA) constitute another approach to introduce money into growth models. In such models the role of money in facilitating transactions is identified by the simple rule that every transaction requires money, and that it must therefore be held for some time in advance. The simplest model, which parallels Sidrauski’s (1967b) analysis, was introduced into the monetary growth literature by Stockman (1981).

\(^5\)Infrastructure investments can be interpreted as investments in maintenance of the transport system or administration, while human capital investments include education and training.

\(^6\)The assumption of CRS becomes much more plausible when capital is viewed broadly to encompass human capital.
As in the Lucasian two-sector growth model, the engine of endogenous growth is the accumulation of human capital. Our model makes use of the same accumulation mechanism,

\[ \dot{h} = \delta(\cdot)(1 - u)h, \]

where Lucas’ (1988) efficiency parameter $\delta$ is now a function of the per–capita government expenditures for human capital formation as in Barro (1990):

\[ \delta(\cdot) = \delta_0 \left( \frac{g_H}{h} \right)^{\delta_1}. \]

Thus the human production technology is given by

\[ \dot{h} = \delta_0 (1 - u)g_H^{\delta_1} h^{1-\delta_1}, \]

where $(1 - u)$ denotes the time devoted to the reproduction of human capital. The production function (5) exhibits constant return to scale in $g_H$ and $h$, but diminishing returns to $h$ separately. That is, even with a broad concept of private capital, production involves decreasing returns to private inputs if the complementary government inputs are not expanded in a parallel manner.

For studying steady-state growth, however, the important fact is CRS in the factors that can be accumulated, which means the two inputs taken together, and not distinction between inputs in human capital production technology.

Hence, the per capita budget-constraint of economy is defined by

\[ c(t) + g(t) + \dot{k}(t) = A k^{\beta_1} g_H^{\beta_2} (uh)^{1-\beta_1-\beta_2}, \]

with $c(t)$ denoting per capita consumption and $\dot{k}(t)$ per capita net investment.

At each point in time real per capita non-human wealth, $w$, is allocated between capital $k$ and real cash balances $m$:

\[ w - k - m = 0. \]

Since $w$ is real per capita non-human wealth, it evolves through time according to

\[ \dot{w} = y(k, uh, g_k) - c - \pi m - \tau, \]

where $\tau$ is the per capita lump-sum tax and $\pi$ is the rate of inflation. As usual, we assume perfect foresight and we will therefore make no distinction between expected and actual inflation.

### 2.3 Government Budget Constraint

Government expenditures $g = g_k + g_h$ are allocated to the production of physical and human capital, where the efficiency of government investments are measured by $\beta_2$ and $\delta_1$.

Since we are reflecting upon the long-run implications of monetary and fiscal policies, we ignore the possibility of government borrowing as justified by Sargent & Wallace (1981) and Sargent (1987).

The government budget constraint is

\[ g_k + g_h = \tau + m\mu, \]

\[ \text{Following convention, the dot notation is used for the time derivative } \dot{x} \equiv \frac{dx}{dt}, \text{ while } \dot{x} \equiv \frac{d}{dt} \text{ denotes the growth rate of the variable } x(t). \]
where $\tau$ is the per-capita tax and $m^{\mu}$ is the seignorage, which is defined as the amount of real resources bought by the government by means of new base money creation following Cukierman (1992):

$$\frac{M}{P} = \mu m = \pi m + \dot{m},$$  \hspace{1cm} (10)

where $M$ is the per-capita money supply, $p$ is the price level and $m$ is the real per-capita money demand.

Since there is no income taxation, government expenditures have to be financed either by taxes or money creation. This assumption is made only for simplicity and could be easily relaxed. Some of the results of the papers are concerned with equivalence of (allocating neutral) poll taxes and inflation taxes. In terms of public finance the identity (10) can be interpreted as an inflation tax with the tax rate $\mu$ and the tax base $m$.

### 3 The Competitive Equilibrium

After the description of the environment we can define our optimal control problem and assume that the representative agent maximizes the utility by solving the following intertemporal maximization problem:

$$\max \int_0^\infty e^{-\rho t} \left\{ \frac{[(1-\alpha)c(t)^\varepsilon + \alpha m(t)^\varepsilon]^\frac{1-\alpha}{1-\varepsilon}}{1 - \sigma} \right\} dt,$$

subject to

$$\dot{w} = Ak^{\beta_1}g_k^{\beta_2} [uh]^{1-\beta_1-\beta_2} - c - \tau - m\pi$$
$$\dot{h} = \delta_0 (1-u)g_h^{\beta_1}h^{1-\delta_1}$$
$$w = k + m$$
$$k(0) = k_0 > 0$$
$$h(0) = h_0 > 0$$
$$u \in [0,1].$$

Individuals choose a stream of consumption $c$, the real balance of money for transactions and the proportion of time they want to spend working $u$ as opposed to studying $(1-u)$. Given the initial values of physical and human capital, the representative agent maximizes utility subject to the specified constraints (8), (5) and (7). To solve the model, we set up the corresponding current value Hamiltonian

$$H = \frac{1}{1 - \sigma} [(1-\alpha)c(t)^\varepsilon + \alpha m(t)^\varepsilon]^{\frac{1-\alpha}{1-\varepsilon}} + \Theta_1 [Ak^{\beta_1}g_k^{\beta_2} (uh)^{1-\beta_1-\beta_2} - c - \pi m - \tau] + \Theta_2 [\delta_0 (1-u)g_h^{\beta_1}h^{1-\delta_1}] + \lambda [w - k - m],$$

where $\Theta_1$ and $\Theta_2$ are the costate variables corresponding to the flow constraints $\dot{w}$ and $\dot{h}$ and $\lambda$ denotes the Lagrangian multiplier associated with the asset constraint.

The necessary conditions (FOC) to our problem with respect to $c$, $m$, $u$, $k$ and $\lambda$ are, respectively,

$$U_c - \Theta_1 = 0$$
$$U_m - \Theta_1 \pi - \lambda = 0$$
$$\Theta_1 f_u - \Theta_2 h_u = 0$$
$$\Theta_1 f_k - \lambda = 0$$
$$w - k - m = 0.$$
Substitution of $\Theta_1$ with $U_c$ from the equation (11) and $\lambda$ with $U_c f_k$ from the equation (14) into the equation (12) gives:

$$\frac{U_m}{U_c} = \pi + f_k = i,$$

where $i$ is the nominal interest rate (real rate of return on capital plus inflation), which can be interpreted as the opportunity cost of holding money or the depreciation rate of the real value of money holdings and which is equal to the marginal rate of substitution between consumption and real money balance.

As long as time can be allocated between employment ("working") and accumulation of human capital ("studying"), the equation (13) implies that the value of marginal unit of time devoted to study must be equal to the value of marginal unit of time devoted to physical production.

Two necessary conditions for the costate variables of the two flow constraints $w$ and $h$ are given by

$$\dot{\Theta}_1 = \rho \Theta_1 - \frac{\partial H}{\partial w} = \rho \Theta_1 - \lambda = \rho \Theta_1 - f_k \Theta_1,$$

$$\dot{\Theta}_2 = \rho \Theta_2 - \frac{\partial H}{\partial h} = \rho \Theta_2 - (1 - \beta_1 - \beta_2) \frac{\gamma}{h} \Theta_1 - \delta_0 \delta_1 (1 - u) \beta_H h^{-\delta_1} \Theta_2,$$

and the following transversality conditions$^8$ must also be met:

$$\lim_{t \to \infty} \left[ e^{-\rho t} \Theta_1(t) k(t) \right] = \lim_{t \to \infty} \left[ e^{-\rho t} \Theta_2(t) h(t) \right] = 0.$$

We will start the analysis of the optimality problem by deriving the money demand function. The necessary condition (16) for household optimality can be solved uniquely for $m = \frac{M^t}{p}$:

$$m = \kappa(i, c) = \left[ \frac{\alpha}{1 - \alpha} \frac{1}{i} \right]^{\frac{1}{1 - \epsilon}} c,$$

which involves only the real transaction volume $c$ and the nominal interest rate $i = f_k + \pi$. For $\epsilon < 1$, $m$ is linear homogeneous in $c$ and decreasing in $i$ as in standard macroeconomic money demand functions.

Money market equilibrium requires that

$$\kappa(i, c) = \frac{M^t}{p}.$$

For a given amount of outside money $M$, a predetermined consumption $c$ and an interest rate $i$, equation (21) determines the price level $p$ of our economy$^9$:

$^8$The sufficient condition, which is well known as Arrow’s condition, for the solution of the FOCs to solve such a problem is that the current value Hamiltonian must be jointly concave in $(k, h)$ after the controls $c, m$ and $u$ have been substituted by their maximizing values. Without any detailed analysis here we have found that $H^*$ is linear in $k$ and $h$,

$$H^* = Z_1 + Z_2 Ak + Z_3 h,$$

where the $Z_i$ are terms excluding $k$ and $h$. Thus, the maximized Hamiltonian $H^*$ is clearly concave.

$^9$For reasons of simplicity we assume that the monetary authority stabilizes the nominal interest rate on its steady-state level during the transition process,

$$\mu(t) = i^* + \dot{c}(t) - f_k(t)$$

$$= i^* + \frac{1}{\sigma} [(1 - \sigma) f_k(t) - \rho].$$

Using the steady-state relationship $i^* = \pi^* + f_k^* = \mu^* + \rho - (1 - \sigma) \psi$ we get the following adjustment rule concerning the growth of money supply compatible with a stabilized nominal interest rate regime:

$$\mu(t) - \mu^* = \frac{1 - \sigma}{\sigma} (f_k(t) - f_k^*)$$

$$= \frac{1 - \sigma}{\sigma} (f_k(t) - \rho - \sigma \psi).$$
Our results can be summarized to the following definition, which closes this section.

**Definition 1 (Competitive Equilibrium).** A competitive equilibrium of the model is a set of sequences for \( c(t), u(t), m(t), k(t), h(t), g(t), p(t) \) and \( g_k, g_h, \tau, \mu \), for \( t \geq 0 \) which satisfy

i. the first order conditions (11)–(15),(17),(18) and the transversality condition (19) of the agents’ maximization problem,

ii. the budget constraints

\[
\dot{k} = A k^{\beta_1} g_k^{\beta_2} (uh)^{1-\beta_1-\beta_2} - c - \pi m - \tau - \dot{m} \tag{23}
\]
\[
\dot{h} = \delta_0 (1-u) g_h^{\delta_1} h^{1-\delta_1} \tag{24}
\]
\[
g = g_k + g_h = \tau + m \mu, \tag{25}
\]

iii. the market clearing conditions

\[
y = \dot{k} + c + g \tag{26}
\]
\[
\frac{M}{p} = m. \tag{27}
\]

Then, the evolution of human capital \( h \), physical capital \( k \), consumption \( c \) and the fraction of time devoted to physical work \( u \) can be considered in the dynamic system (28)–(31) as shown in the appendix:

\[
\dot{c} = \frac{\beta_1 A}{\sigma} k^{\beta_1-1} u^{1-\beta_1-\beta_2} h^{1-\beta_1-\beta_2} g_k^{\beta_2} c - \frac{\rho}{c} \tag{28}
\]
\[
\dot{u} = \frac{\delta_0 (1-(1-u)\delta_1)}{\beta_1 + \beta_2} u - \frac{\beta_1}{\beta_1 + \beta_2} \left( \frac{c}{k} + \frac{g}{k} \right) u - \frac{\beta_1}{\beta_1 + \beta_2} \delta_0 (1-u) g_h^{\delta_1} h^{1-\delta_1} u \tag{29}
\]
\[
\dot{k} = A k^{\beta_1} g_k^{\beta_2} u^{1-\beta_1-\beta_2} h^{1-\beta_1-\beta_2} - c \tag{30}
\]
\[
\dot{h} = \delta_0 (1-u) g_h^{\delta_1} h^{1-\delta_1}. \tag{31}
\]

**4 The Balanced Growth Path**

The dynamics of our model are described by the system of nonlinear first order differential equations (28)-(31). In order to derive the steady-state equilibrium of the model we transfer the four-dimensional system into a saddle–point–stable autonomous three-dimensional differential equation system by a change of variables as in Mulligan & Sala-i-Martin (1993) and Benhabib & Perli (1994).

**Definition 2 (Balanced Growth Path).** A balanced growth path (or steady-state equilibrium) is a competitive equilibrium \( \{c(t), k(t), h(t), u(t)\} \) with initial conditions \( k(0) = k_0, h(0) = h_0 \), such that \( c(t), k(t) \) and \( h(t) \) are growing at constant rates, \( u(t) \) is constant, and the government expenditure ratios \( \frac{g_k}{k} \) and \( \frac{g_h}{h} \) are constant.

Next we can define the following new stationary variables:

\[
\tilde{c} = \frac{c}{h}, \quad \tilde{c} = \tilde{c} - \dot{h} \tag{32}
\]
\( \tilde{k} = \frac{k}{h}, \quad \ddot{k} = \ddot{k} - \dot{h} \quad (33) \)
\( \ddot{g} = \frac{g}{h}, \quad g_k = vg, \quad g_h = (1 - v)g \quad (34) \)
\( \ddot{y} = \frac{y}{h}, \quad \ddot{y} = A k^{\beta_1} [\nu g]^{\beta_2} u^{1-\beta_1-\beta_2}. \quad (35) \)

By taking logarithms in equation (32) and (33), differentiating with respect to time, and substituting for \( \dot{c}, \dot{h} \) and \( \ddot{k} \), we get expressions for \( \ddot{c} \) and \( \ddot{k} \) and with the dynamic equation (29), we obtain the following three-dimensional system:

\[
\ddot{c} = \frac{\beta_1 A}{\sigma} v^{\beta_2} k^{\beta_1-1} u^{1-\beta_1-\beta_2} g^{\beta_2} - \frac{\rho}{\sigma} - \delta_0 (1 - \delta_1)(1 - u)(1 - v) \delta_1 \ddot{g} \delta_1 \quad (36)
\]

\[
\ddot{k} = A v^{\beta_2} k^{\beta_1-1} u^{1-\beta_1-\beta_2} g^{\beta_2} - \left( \frac{\ddot{c}}{k} + \frac{\ddot{g}}{k} \right) - \delta_0 (1 - u)(1 - v) \delta_1 \ddot{g} \delta_1 \quad (37)
\]

\[
\ddot{u} = \frac{1}{\beta_1 + \beta_2} \delta_0 (1 - (1 - u)) \delta_1 (1 - v) \delta_1 \ddot{g} \delta_1 - \frac{\beta_1}{\beta_1 + \beta_2} \left( \frac{\ddot{c}}{k} + \frac{\ddot{g}}{k} \right) \quad (38)
\]

\[
- \frac{\beta_1}{\beta_1 + \beta_2} \delta_0 (1 - u)(1 - v) \delta_1 \ddot{g} \delta_1.
\]

The variables \( k, h \) and \( c \) do not appear in equations (36 – 38), which means that we transfer the previous four-dimensional system into a system with only three dimensions, namely in \( \ddot{c}, \ddot{k} \) and \( u \).

In order to characterize the steady-state of the reduced model, we set the growth rates of \( \ddot{k}, \ddot{c} \) and \( \ddot{u} \) equal to zero, \( \ddot{k} = \ddot{c} = \ddot{u} = 0 \). After some algebraic manipulations we obtain for the steady state the following equation system:

\[
\dot{k} = \frac{\ddot{y}}{k} - \frac{\ddot{c} + \ddot{g}_k + \ddot{g}_h}{k} - \delta_0 \delta_1 (1 - u) = 0 \quad (39)
\]

\[
\dot{c} = \sigma^{-1} \left( \frac{\beta_1}{k} \ddot{y} - \rho \right) - \delta_0 \delta_1 (1 - u) = 0 \quad (40)
\]

\[
\dot{u} = - \frac{\ddot{c} + \ddot{g}_k + \ddot{g}_h}{k} - \frac{\delta_1 + \beta_1}{\beta_1} \delta_0 \delta_1 (1 - u) + \beta_1^{-1} \delta_1 = 0, \quad (41)
\]

where \( \ddot{y} = k^{\beta_1} g_k u^{1-\beta_1-\beta_2} \). We see immediately that this system is linear in \( \frac{\ddot{y}}{k}, \frac{\ddot{c} + \ddot{g}_k + \ddot{g}_h}{k} \) and \( u \):

\[
\left( \frac{\ddot{y}}{k} \right)^* = \frac{\rho \delta_1 + \sigma \delta_0 \delta_1}{\beta_1 (\delta_1 + \sigma)} \quad (42)
\]

\[
\left( \frac{\ddot{c} + \ddot{g}_k + \ddot{g}_h}{k} \right)^* = \frac{\rho}{\beta_1} + \frac{\sigma - \beta_1 (\delta_0 \delta_1 - \rho)}{\beta_1 (\delta_1 + \sigma)} \quad (43)
\]

\[
u^* = 1 - \frac{1}{\delta_1 + \sigma} \left[ 1 - \rho \delta_0^{-1} g^{-\delta_1} \right]. \quad (44)
\]

This observation allows the explicit solution of this equation system for the steady-state values of the model:

\[
\tilde{k}^* = A^{\frac{1}{1-\beta_1}} \frac{\partial_2}{\dot{g}_k} \left( \frac{\beta_1 (\delta_1 + \sigma)}{\rho (\delta_1 + \sigma) + \sigma (\delta_0 \delta_1 - \rho)} \right) \frac{1}{(\delta_1 + 1)\frac{1}{\delta_1} + \frac{\partial_1}{\delta_1} + \frac{\delta_1}{\rho (\delta_1 + \sigma) + \sigma (\delta_0 \delta_1 - \rho)}} \quad (45)
\]

\[
\tilde{c}^* = \frac{\delta_1 + \sigma + (\delta_0 \delta_1 - \rho)}{\beta_1 (\delta_1 + \sigma)} \frac{\partial_2}{\dot{g}_k} \left( \frac{\beta_1 (\delta_1 + \sigma)}{\rho (\delta_1 + \sigma) + \sigma (\delta_0 \delta_1 - \rho)} \right) \quad (46)
\]
\[ u^* = 1 - \frac{\delta_0 \tilde{g}_h^\delta_1 - \rho}{(\delta_1 + \sigma) \delta_0 \tilde{g}_h^\delta_1} \]

Equations (45), (46) and (47) represent the steady state of the reduced model.

Now we can derive the steady-state growth rate of human capital production, \( \hat{h}^* = \delta_0 \tilde{g}_h^\delta_1 (1 - u^*) \), which is equal to the steady-state growth rate of physical capital, income and consumption:

\[ c^* = \hat{y}^* = \hat{k}^* = \frac{\delta_0 \tilde{g}_h^\delta_1 - \rho}{\delta_1 + \sigma} =: \psi. \]

Equation (48) shows how the steady-state growth rate is determined endogenously by taste parameters \( \sigma, \rho \) and technology parameters \( \delta_0, \delta_1 \). One immediately recognizes that the steady-state growth rate is positively related to the government spending in human capital production sector \( \tilde{g}_h \). The government spending on physical production sector \( \tilde{g}_k \) has no effects on long-run growth.

The following compact representation of the steady-state values results:

\[ u^* = 1 - \frac{\psi}{\delta_0 \tilde{g}_h^\delta_1}, \]

\[ \tilde{k}^* = \left( A \frac{1}{1 + \rho} \left( \frac{\beta_1}{\beta_1} \right)^{1 - \rho \psi} \left( 1 - \frac{\psi}{\delta_0 \tilde{g}_h^\delta_1} \right)^{1 - \rho \psi} \right), \]

\[ c^* = \frac{\rho + (\sigma - \beta_1) \psi}{\beta_1} \tilde{k}^* - \tilde{g}_h - \tilde{g}_k. \]

Our results are based on the assumption that the human capital production sector exhibits constant returns to scale, when the accumulable factor \( h \) and the government expenditures are adjusted simultaneously.\(^{10}\)

The monetary sector of the economy behaves only as an appendix to the dynamics in the real economy considered above. Money market equilibrium implies in the steady-state that the rate of inflation \( \pi = \frac{\dot{p}}{p} \) equals the monetary expansion rate \( \mu = \frac{M}{M} \) minus the fundamental rate of growth \( \psi \):

\[ \pi = \mu - \psi. \]

4.1 Existence

**Proposition 1 (Existence).** Given that the parameter vector \( \omega = \{A, \beta_1, \beta_2, \delta_0, \delta_1, \rho, \sigma\} \) lies in the subset \( \Omega^* \) of the parameter space \( \Omega \),

\[ \Omega^* = \left\{ \omega \in \Omega \mid x > \rho, \sigma > 1 - \frac{\rho}{x} [1 + x \delta_1] \right\} \text{ and } \sigma > \beta_1 \]

there exists a steady-state equilibrium where the growth rate of income, physical and human capital is given by:

\[ \Psi = \frac{x - \rho}{\varphi}, \]

\(^{10}\)Notice that if one dismisses government spending in the human capital production sector, \( \delta_1 = 0 \), the following steady-state growth rate ensures:

\[ \psi = \frac{\delta_0 - \rho}{\sigma}. \]

Interestingly, this growth rate is exactly the same as in Lucas’ model (1988) if the externality does not appear, i.e. \( \gamma = 0 \).
where \( x = \delta_0 \hat{g}_h^{\delta_1} \) and \( \varphi = (\delta_1 + \sigma) \). The restrictions on the parameter space finally guarantee that \( u^* \in (0, 1), c^* > 0, \tilde{k}^* > 0 \) and \( \tilde{y}^* > 0 \).

**Proof.** See the Appendix.

The following compact representation of the steady-state values results:

\[
\begin{align*}
    u^* &= 1 - \frac{\Psi}{\delta_0 \hat{g}_h^{\delta_1}}, \\
    \tilde{k}^* &= A \frac{1 - \beta_2}{1 - \beta_1} \hat{g}_h^{\beta_2} \left( \frac{\beta_1}{\rho + \sigma \Psi} \right)^{\frac{1}{1 - \beta_1}} \left( 1 - \frac{\Psi}{\delta_0 \hat{g}_h^{\delta_1}} \right)^{\frac{1 - \beta_2}{1 - \beta_1}}, \\
    \tilde{c}^* &= \frac{\rho + (\sigma - \beta_1) \Psi}{\beta_1} \tilde{k}^* - \tilde{g}.
\end{align*}
\]

Our results are based on the assumption that the human-capital production sector exhibits constant returns to scale, when the accumulable factor \( h \) and the government expenditures are adjusted simultaneously.

Notice that if one dismisses government spending in the human-capital production sector, \( \delta_1 = 0 \), the following steady-state growth rate ensures:

\[
\Psi = \frac{\delta_0 - \rho}{\sigma}.
\]

Interestingly, this growth rate is exactly the same as in Lucas’ model (1988) if the externality does not appear, i.e. \( \gamma = 0 \).

As noted above, if \( \omega \in \Omega \), then we can be sure that there exists a steady state. Now, to complete the steady-state analysis we have to check whether the steady state satisfies the transversality conditions.

The transversality condition for physical capital described by equation (19) to be satisfied it must be that

\[
\lim_{t \to \infty} \left[ -\rho + \frac{\dot{\Theta}_1}{\Theta_1} + \frac{\dot{k}}{k} \right] < 0. \tag{53}
\]

Substituting from equation (17) for \( \dot{\Theta}_1 / \Theta_1 \) and equation (30) for \( \dot{k}/k \) the above transversality condition is reduced to

\[
(1 - \beta_1)(1 - \tau_y) \hat{y}_k - \left( \frac{\tilde{c}}{k} + \frac{\tilde{r}}{k} \right) < 0. \tag{54}
\]

If \( u \) is in the unit interval this guarantees that the above expression is always satisfied. As for the transversality condition for human capital, given by equation (19), we have

\[
\lim_{t \to \infty} \left[ -\rho + \frac{\dot{\Theta}_2}{\Theta_2} + \frac{\dot{h}}{h} \right] < 0. \tag{55}
\]

Substituting from equation (18) for \( \dot{\Theta}_2 / \Theta_2 \) and equation (5) for \( \dot{h}/h \) the above transversality condition is reduced to

\[
-ux < 0. \tag{56}
\]

Thus, the second transversality condition is always satisfied, provided that \( u \) lies in the unit interval.

Hence, we can be sure that the solution described above is a maximum, since the Hamiltonian is concave, and both terminal conditions are satisfied at the steady state.
4.2 Local Stability

**Proposition 2 (Stability).** Let $\omega \in \Omega$, then the equilibrium is locally unique. The Jacobian evaluated at the balanced growth path $J^*$ has one negative eigenvalue and two eigenvalues with positive real parts.

**Proof.** See for the extended Proof Woehrmann (1998).

Note that Benhabib & Perli (1994) and Mulligan & Sala-i-Martin (1993), in studying the Lucas model with externalities, find saddle-path stability, consistent with Proposition 1 above, for slightly modified parameter restrictions, because they did not consider the case with government investments. But it is worth noting that if we assume a case, where $x < \delta < \rho$, there can be a continuum of solutions in the neighbourhood of the BGP, which means the system will be subject to indeterminacy problems.

Generally, in a number of models found in the literature, indeterminacy is associated with stable dynamics around a steady state (see, e.g. Benhabib & Farmer, 1994, 1997). In that case, there is a continuum of trajectories (all of which converge to the steady state) which satisfy all the equilibrium conditions (including the transversality conditions). It is beyond the scope of this work to discuss the recent growth models with indeterminacy or to investigate the conditions under which our basic model may exhibit indeterminacy. The author refers readers to Benhabib & Gali (1995), who have written a masterful survey of economic growth models with indeterminacy.

This section has been restricted to the characterization of the steady-state equilibrium. In the next section we consider the role of monetary and fiscal policies along the balanced growth path. Generally, the analysis of the transitional dynamics merits further investigation.

5 Monetary and Fiscal Policies

In the following we will examine how monetary and fiscal policies influence the accumulation of human and physical capital. We start with a study of the effects of a permanent increase in the money supply which will be characterised by a neutrality proposition. Then we address the issue of whether or not real variables such as human and physical capital, time allocation and consumption are dependent upon the rate of monetary growth. In the steady-state, the superneutrality of money depends on the reaction of $\Psi$ to an increase in the monetary growth rate $\mu$. Concerning the long-run rate of inflation, an invariant real growth rate would imply the quantity-theoretical result that $d\pi = d\mu$. Finally, we focus on the interdependence of monetary and fiscal policy based on government budget constraint as emphasised by Sargent & Wallace (1981) in their unpleasant monetarist arithmetic.\[11\]

5.1 Neutrality of Money

Equation (22) gives rise to the following proposition, which can be seen as a consistency condition for our flex-price model under consideration.

**Proposition 3 (Neutrality of Money).** An increase in the per-capita money supply $dM \geq 0$ leads to an immediate increase in the price level $p$ and leaves all real variables unaltered. If the monetary expansion is permanent, $dM(t) = \nu M(t), t \geq 0$, the increase in the price level is equiproporional $dp(t) = \nu p(t), t \geq 0$. The (expected) rate of inflation remains unchanged as well as the seignorage.

**Proof.** The proof is standard.

\[11\] The ability of government to be alimented by the monetary authority is often restricted by various institutional devices. Nevertheless, there are enormous cross-country differences in the extent to which the government is allowed to borrow from the central bank. In developing countries the government is often entitled to borrow directly from the central bank and it exploits this privilege to finance a non-negligible fraction of its expenditures. As stressed by Cukierman (1992), this behaviour might be explained by their imperfect and rudimentary tax systems and the relatively narrow capital markets.
5.2 Steady-State Analysis

The governmental right to create currency has been used as an instrument for financing government expenditures ever since the exchange value of money exceeded its intrinsic value. Hence the examination of the budgetary implications of an accelerated monetary growth has a long tradition in macroeconomics and is considered now:

Lemma 1 (Seignorage). Suppose the sequence of per-capita government spending on human-capital production $g_h$ is given. Then, the steady–state ratio of government spending financed by seignorage is an increasing function of the growth rate of the money supply, if and only if $\rho - (1 - \sigma)\Psi > -\epsilon\mu$. A sufficient condition is a positive rate of monetary expansion.

The steady-state seignorage $s$ as a function of the money supply growth rate $\mu$ and the parameter $\epsilon$ of the money-in-the-utility-function is illustrated in figure 1. In the long run a budget deficit has to be monetarized (as suggested by Sargent & Wallace (1988). Thus, in the long run, a balanced budget is a necessary condition to achieve price stability.

Proposition 4 (Superneutrality of Money). If in addition the financial revenues from money creation are used to lower taxation $d\tau(\mu) = d\tau$, consumption allocation remains unaltered. Hence, steady-state consumption is also indifferent with respect to the choice of fiscal instruments.

Proof. For a given sequence of government expenditures $g$, an increase in the monetary expansion must coincide with lower taxation. From equations (28) to (31) it follows immediately that the real steady-state variables $c^*$, $k^*$ and $y^*$ depend neither on lump-sum taxes nor on the rate of money creation. Thus these magnitudes are unaltered by an increase in the money supply. According to equations (48) and (52) the inflation rate increases, $\pi = \mu - \bar{\psi}$. For a given real interest rate $f_k^*$ the nominal interest rate, $i = f_k^* + \pi^*$, increases too. Since the demand for real money balances depends on the nominal interest...
This proposition illustrates the equivalence of financing government expenditures by (poll) taxes and by money creation. While the level of government expenditures \( g \) affects consumption allocation, the way in which they are financed does not.

But as in our model the new theories of endogenous growth do not allow macroeconomic (fiscal and monetary) policies to affect the growth rate of the economy, however, which seems a major discrepancy from the macroeconomists point of view. This is because the endogenous growth theories are based on the model of Ramsey, which assumes infinitely-lived households or dynasties with a fully operative intergenerational bequest motive and therefore implies *Ricardian debt neutrality*. This seems an unnecessarily restrictive framework for addressing questions about the effectiveness of fiscal and monetary policy, and in this sense, the superneutrality result is not a surprise. However, if we extend endogenous growth models to allow for non-interconnected overlapping generations of households, increases in monetary growth accompanied by lump-sum taxes (-transfers) have real effects. These forms of non-neutrality occur only if there is a positive birth rate, i.e. if either agents have finite lives or there is a positive rate of population growth.

Two assumptions are important for delivering the superneutrality proposition: first, the assumption that lump-sum taxes are used rather than distorting taxes; and second, the assumption that the revenues from money creation are not used for investments in human capital production. The latter assumption, under which the neutrality proposition has been derived, will now be replaced by the following: Suppose that a constant fraction \( 1 - v \) of government expenditures is invested in the production of human capital, while a fraction \( v \) of government investments goes into the production of physical capital. Since we know that such expenditures accelerate economic growth even in the long run,

\[
\frac{d\psi}{d\bar{g}_h} = \frac{\delta_1\delta_0}{\sigma + \delta_1}\bar{g}_h^{\delta_1-1},
\]

an increase in government expenditures in the human capital production function alimented by the monetary authority has real effect. This result is stated in our final proposition:

**Proposition 5 (Alimentary Monetary Policy).** *An increase in human capital related government expenditures alimented by the monetary authority has a positive impact on the steady-state growth rate of the economy.*

**Proof.** The proof is in the appendix.

This result shows the monetary policy can be used to enhance growth in the development process of a country if it is the right way. More precisely, the policy makers must used the additional revenues for investments in production sectors, where production exhibits increasing returns to scale. In our model, the human capital sector exhibits increasing returns, but this result can be used also to any industries with increasing returns such as information industry, biotechnology industry etc.

In the recent political discussion, the policy makers from developing countries and eastern europe are often considering monetary policy as only an effective policy for their financial crisis. But as this paper proved, governments must care attention if they use the monetary policy as stabilizing instrument because of their expenditure effects.

These results provide some interesting insights into the debate on the real effects of monetary poicy. Our final proposition predicts that empirically observed correlations between monetary growth rates

12*Ricardian debt neutrality, which says that it does not matter whether a given stream of government spending is finance through current taxes or through future taxes made possible through the issue of bonds, and superneutrality of money.*
and real economic growth depend on the allocation of government expenditures financed by seignorage. This is in line with the findings in econometric literature that there is no stable relation between money and growth.

6 Conclusions

The purpose of this paper was to integrate money in Lucas’ (1988) two-sectorial endogenous growth model. Developing this monetary version of Lucas’ general equilibrium model, we have provided a general analysis of the links between money and economic growth. It was shown that the effects of macroeconomic policies critically depend on the allocation on government expenditures. Long-run growth is based on the intentional accumulation of human capital. Hence it is affected only by government investments in human capital (schooling), not by expenditures accelerating market production (investments in the technical infrastructure).

Main attention was given to model the interdependence of fiscal and monetary policy due to the government budget constraint. The opportunity of financing allocative relevant government expenditures by seignorage gives rise to a relaxation of the predictions based on superneutrality propositions. In contrast to the standard neoclassical monetary growth model, an increase in the rate of money supply growth can have real effects.

7 Appendix

7.1 Dynamic System

In this appendix we derive the system of differential equations (28)-(31).

The explicit first-order conditions to the optimization problem with respect to $c$, $m$, $u$, $k$, $w$ and $h$ are:

\[
U_c = \frac{1}{1 - \sigma}((1 - \alpha)c^\epsilon + \alpha m^\epsilon)^{\frac{1 - \sigma}{\epsilon} - 1}(1 - \alpha)c^{\epsilon - 1} = \Theta_1 \tag{G.1}
\]

\[
U_m = \frac{1}{1 - \sigma}((1 - \alpha)c^\epsilon + \alpha m^\epsilon)^{\frac{1 - \sigma}{\epsilon} - 1}\alpha m^{\epsilon - 1} = \Theta_1 \pi + \lambda \tag{G.2}
\]

\[
\Theta_2 \delta_0 h = \Theta_1 Ak^{\beta_1}(1 - \beta_1 - \beta_2)h^{1 - \beta_1 - \beta_2 - \delta_1 u^\beta_1 - \beta_2 g_k} \tag{G.3}
\]

\[
\lambda = \beta_1 Ak^{\beta_1 - 1}g_k^{\beta_2}u^{1 - \beta_1 - \beta_2}h^{1 - \beta_1 - \beta_2} \Theta_1 \tag{G.4}
\]

\[
w = k + m \tag{G.5}
\]

\[
\dot{\Theta}_1 = \rho \Theta_1 - \beta_1 Ak^{\beta_1 - 1}g_k^{\beta_2}u^{1 - \beta_1 - \beta_2}h^{1 - \beta_1 - \beta_2} \Theta_1 \tag{G.6}
\]

\[
\dot{\Theta}_2 = \rho \Theta_2 - A(1 - \beta_1 - \beta_2)k^{\beta_1}g_k^{\beta_2}u^{1 - \beta_1 - \beta_2}h^{-\beta_1 - \beta_2} \Theta_1 \tag{G.7}
\]

\[-(1 - \delta_1)\delta_0(1 - u)g_h^{\beta_1}h^{-\beta_1} \Theta_2 , \tag{G.8}
\]

where the consistency conditions $g = g_k + g_h = \bar{m} + \pi m + \tau$ have been used. The two limiting transversality conditions are

\[
\lim_{t \to \infty} [e^{-\rho t} \Theta_1(t)k(t)] = \lim_{t \to \infty} [e^{-\rho t} \Theta_2(t)h(t)] = 0. \tag{G.9}
\]

Solving equations (11) and (12) for the optimal money demand as in equation (20) and inserting the resulting money demand function into equation (11) gives

\[
\Theta_1 = Zc^{-\sigma}
\]
where \( Z \equiv \left\{ (1 - \alpha)[(1 - \alpha) + \alpha \kappa(i, 1)^{1 - \sigma - \epsilon}] \right\} \).

From (17) we have,
\[
\dot{c} = \sigma^{-1}[\beta_1 A k^{\beta_1 - 1} g_k^{\beta_2} u^{1 - \beta_1 - \beta_2} h^{1 - \beta_1 - \beta_2} - \rho].
\]
Substituting for \( \Theta_1 \) in (13), we derive an expression for \( \Theta_2 \):
\[
\Theta_2 = \frac{A(1 - \beta_1 - \beta_2)}{\delta_0} k^{\beta_1} g_k^{\beta_1 - \delta_1} g_k^{\beta_2} h^{\beta_1 - \beta_2 - 1} u^{-\beta_1 - \beta_2} \Theta_1. \tag{G.10}
\]
Under the assumption that \( \frac{\delta_k}{\delta_h} = \tilde{g}_k \) and \( \frac{\delta_h}{\delta_h} = \tilde{g}_h \) are always constant we can express the above equation (G.10) as:
\[
\Theta_2 = \frac{A(1 - \beta_1 - \beta_2)}{\delta_0} Z k^{\beta_1} g_h^{\beta_2} h^{\beta_1 - \beta_2 - 1} u^{-\beta_1 - \beta_2} c^{-\sigma}. \tag{G.11}
\]
Now take logarithms and time derivatives of both sides of (G.11):
\[
\dot{\Theta}_2 = \beta_1 \dot{k} - \beta_1 \dot{h} - \sigma \dot{c} - (\beta_1 + \beta_2) \dot{u}. \tag{G.12}
\]
We can substitute the rates of \( k, h, \) and \( c \) into (G.12) to get an expression for the growth rate of the shadow price of human capital:
\[
\dot{\Theta}_2 = \rho - \beta_1 \left( \frac{\ddot{c} + \tilde{g}_k + \tilde{g}_h}{k} \right) - \beta_1 \delta_0 (1 - u) g_h^{\delta_1} h^{-\delta_1} - (\beta_1 + \beta_2) \dot{u}. \tag{G.13}
\]
Next we derive the rate of growth of \( \Theta_2 \) from (G.7) and set it equal to the expression in equation (G.13),
\[
\rho - \beta_1 \left( \frac{\ddot{c} + \tilde{g}_k + \tilde{g}_h}{k} \right) - \beta_1 \delta_0 (1 - u) g_h^{\delta_1} h^{-\delta_1} - (\beta_1 + \beta_2) \dot{u} = \rho - \delta_0 u g_h^{\delta_1} h^{-\delta_1} - (1 - \delta_1) \delta_0 (1 - u) g_h^{\delta_1} h^{-\delta_1}.
\]
From this equation we can derive an expression for \( \dot{u} \):
\[
\dot{u} = \frac{\delta_0 (1 - (1 - u) \delta_1) g_h^{\delta_1} h^{-\delta_1}}{\beta_1 + \beta_2} u - \frac{\beta_1}{\beta_1 + \beta_2} (\frac{c}{k} + \frac{g}{k}) u - \frac{\beta_1}{\beta_1 + \beta_2} \delta_0 (1 - u) g_h^{\delta_1} h^{-\delta_1} u.
\]
After expressing the two multipliers \( \Theta_1 \) and \( \Theta_2 \) in terms of their corresponding control variables \( c \) and \( u \) and with the rest of the accumulation constraints, we entail an autonomous system of four nonlinear differential equations in these four variables \( (c, u, k, h) \):
\[
\dot{c} = \frac{\beta_1 A}{\sigma} k^{\beta_1 - 1} u^{1 - \beta_1 - \beta_2} h^{1 - \beta_1 - \beta_2} g_k^{\beta_2} c - \frac{\rho}{\sigma} c, \tag{G.14}
\]
\[
\dot{u} = \frac{\delta_0 (1 - (1 - u) \delta_1) g_h^{\delta_1} h^{-\delta_1}}{\beta_1 + \beta_2} u - \frac{\beta_1}{\beta_1 + \beta_2} (\frac{c}{k} + \frac{g}{k}) u - \frac{\beta_1}{\beta_1 + \beta_2} \delta_0 (1 - u) g_h^{\delta_1} h^{-\delta_1} u, \tag{G.15}
\]
\[
\dot{k} = A k^{\beta_1} g_k^{\beta_2} u^{1 - \beta_1 - \beta_2} h^{1 - \beta_1 - \beta_2} - c - g, \tag{G.16}
\]
\[
\dot{h} = \delta_0 (1 - u) g_h^{\delta_1} h^{1 - \delta_1}. \tag{G.17}
\]
7.2 Proof of Proposition 1

In order to characterise the steady state, we set the growth rates of \( \tilde{k}, \tilde{c} \) and \( \tilde{u} \) to zero:

\[
\tilde{k} = \tilde{c} = \tilde{u} = 0.
\]

After some algebraic manipulations we obtain a linear equation system in the steady-state values \( \tilde{y}^*, \tilde{c}^* \) and \( \tilde{u}^* \) allowing the explicit solution of this equation system. Solving the three-dimensional system finally gives the steady-state values of the intensive form of the model and restrictions on the parameter space finally guarantee that \( u^* \in (0, 1), \bar{c} > 0 \) and \( \tilde{k} > 0 \) as can see immediately from the steady solutions (45 – 47). The restrictions on the parameter space finally guarantee that \( u^* \in (0, 1), \bar{c} > 0 \) and \( \tilde{y}^* > 0 \). It is worth is to notice that above parameter restrictions is also valid to guarantee that the steady-state solution (42 – 44) to a system normalised to \( k \). This implies that \( u^* \in (0, 1), c^*/k^* > 0 \) and \( y^*/k^* > 0 \).

7.3 Proof of Proposition 5

Consider again the government budget constraint:

\[
\tilde{g}_h = (1 - v) \left[ \frac{\tau}{h} + \frac{m}{h} \mu \right].
\]

Let us assume that the poll taxes \( \tau \) are indexed to the general skill level \( h \), such that \( \tilde{\tau} = \frac{\tau}{h} \) is constant and an increase in government expenditures is only financed by a higher growth of money supply \( \mu \). A balanced budget via seignorage implies that monetary and fiscal policies are related as follows:

\[
d\tilde{g}_h = d \left\{ (1 - v) \frac{m}{h} \mu \right\}
= (1 - v) d \left\{ \kappa \left( \mu + \rho - (1 - \sigma) \psi, \frac{c}{h} \right) \mu \right\}
= (1 - v) \left\{ \kappa (\mu + \rho - (1 - \sigma) \psi, \tilde{c}) + \frac{\partial \kappa (\mu + \rho - (1 - \sigma) \psi, \tilde{c})}{\partial i} \mu d\psi + \frac{\partial \kappa (\mu + \rho - (1 - \sigma) \psi, \tilde{c})}{\partial c} \mu d\tilde{c} \right\},
\]

where \( d\tilde{c} = [\tilde{c}_g + \tilde{c}_g h \tilde{k}_g + (\tilde{c}_g + \tilde{c}_k h \tilde{g}_g)]d\tilde{g}_h \) and \( d\psi = \tilde{c}_g d\tilde{g}_h \). According to equation (48) the fundamental growth rate \( \psi \), the normalized consumption \( \tilde{c} \) and the capital intensity \( \tilde{k} = \frac{h}{k} \) are defined as follows:

\[
\psi (\tilde{g}_h) = \frac{\delta_0 d_1 - \rho}{\delta_1 + \sigma}, \quad \text{(G.18)}
\]

\[
\tilde{c} (\psi (\tilde{g}_h), \tilde{g}_h, \tilde{k} (\psi (\tilde{g}_h), \tilde{g}_h)) = \frac{\rho + (\sigma - \beta_1) \psi}{\beta_1} \tilde{k} (\psi (\tilde{g}_h), \tilde{g}_h) - \frac{1}{1 - v} \tilde{g}_h, \quad \text{(G.19)}
\]

\[
\tilde{k} (\psi (\tilde{g}_h), \tilde{g}_h) = A \left( \frac{v}{1 - v \tilde{g}_h} \right)^{\frac{\delta_2}{1 - \beta_1}} \left( 1 - \frac{\psi}{\delta_0 d_1} \right)^{\frac{1 - \beta_1 - \beta_2}{1 - \beta_1}} \left( \frac{\rho + \sigma \psi}{\beta_1} \right)^{\frac{1 - \beta_1}{1 - \beta_1}}. \quad \text{(G.20)}
\]

The budgetary relationship between the rate of monetary expansion and government investments in human capital can be summarized to:

\[
\{ 1 + (1 - v) \mu [(1 - \sigma) \kappa \psi \tilde{g}_h - \kappa_c \tilde{c}_g] \} d\tilde{g}_h = (1 - v) [\kappa + \kappa_c \mu] d\mu.
\]
Then, the effect on the long-run growth rate is given by
\[
\frac{d\psi}{d\mu} = \frac{(1 - v)[\kappa + \kappa_i \mu] \psi \tilde{g}_h}{1 + (1 - v)\mu \left\{ (1 - \sigma)\kappa_i - \kappa_c [(\tilde{c}_\psi + \tilde{c}_k \tilde{k}) \psi \tilde{g}_h + (\tilde{c}_\tilde{g}_h + \tilde{c}_k \tilde{g}_h)] \right\}} > 0,
\]
where the long-run multiplier of fiscal policy is
\[
\psi \tilde{g}_h = \frac{\delta_1 \delta_0}{\sigma} \theta_1^{-1} > 0,
\]
and the explicit policy multiplier can be evaluated by using the partial derivatives of the functions (G.18)-(G.20) and the money demand function (20)
\[
\kappa_i = -\frac{1}{1 - \epsilon} \frac{\kappa (\mu + \rho - (1 - \sigma)\psi, \tilde{c})}{\mu + \rho - (1 - \sigma)\psi} < 0,
\]
\[
\kappa_c = \kappa (\mu + \rho - (1 - \sigma)\psi, 1) > 0,
\]
\[
\tilde{c}_\psi = \frac{\sigma - \beta_1}{\beta_1} \tilde{k}(\psi \tilde{g}_h, \tilde{g}_h) > 0,
\]
\[
\tilde{c}_k = \frac{(\sigma - \beta_1)\psi + \rho}{\beta_1} > 0,
\]
\[
\tilde{c}_\tilde{g}_h = -\frac{1}{1 - v} < -1,
\]
\[
\tilde{k}_\psi = \frac{1}{1 - \beta_1} \left[ (1 - \beta_1 - \beta_2) (\psi - \delta_0 \tilde{g}_h^1) - \frac{\sigma}{\rho + \sigma \psi} \right] \tilde{k}(\psi \tilde{g}_h, \tilde{g}_h) < 0,
\]
\[
\tilde{k}_\tilde{g}_h = \left[ \frac{v}{1 - v} \frac{\beta_2}{1 - \beta_1} + \frac{1 - \beta_1 - \beta_2}{\beta_1} \delta_0 \tilde{g}_h^1 \left( \frac{\delta_0 \tilde{g}_h^1}{\delta_0 \tilde{g}_h^1 - \psi} \right) \right] \tilde{k}(\psi \tilde{g}_h, \tilde{g}_h) < 0.
\]
Finally, the effect on the inflation rate is equal to
\[
\frac{d\pi}{d\mu} = 1 - \frac{d\psi}{d\mu}.
\]

References


